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LETTER TO THE EDITOR

A subadditive thermodynamic formalism for mixing repellers

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Abstract. The thermodynamical description of fractals that has recently attracted much interest both experimentally and theoretically in the study of dynamical systems is, in some ways, limited, being essentially an additive theory. We present a subadditive thermodynamic formalism for which we derive a variational principle and show how it may be used to study the dynamics of non-conformal transformations. In particular we discuss an analogue of Bowen's formula for the dimension of a mixing repeller.

The thermodynamical description of fractals has recently attracted considerable interest, partly because of the new concept of 'multifractals' (see Tél (1987), Bohr and Tél (1988), Bessis *et al* (1988) and Vaienti (1988a, b) which include many references). Analogues of concepts such as entropy and pressure may be defined for fractals constructed in a recursive manner, leading to the existence of Gibbs measures on fractals. One of the most elegant results in the area is Bowen's formula which gives the Hausdorff dimension of the repeller of a conformal transformation in terms of the pressure function (Bowen 1979, Ruelle 1983). However, the present thermodynamic formalism is essentially additive, and to enable non-conformal transformations to be included in the framework a subadditive theory is required. Here we establish such a theory, exhibit a variational principle and give a generalisation of Bowen's formula.

Let J be a compact subset of a Riemann manifold M that is invariant under an expanding map f of class $C^{1+\varepsilon}$ (i.e. the derivative of f has Hölder exponent $\varepsilon > 0$). Specifically we assume the following.

(a) There exist $c > 0$ and $\alpha > 1$ such that

$$\|(T_x f^n)(u)\| \geqslant c\alpha^n \|u\| \tag{1}$$

for $x \in J$, $u \in T_x M$ and $n \geqslant 1$, where $T_x f$ is the tangent map to f at x (the derivative if M is $\mathbb{R}^d$).

(b) J has an open neighbourhood V in M such that

$$J = \{x \in V : f^n x \in V \text{ for all } n \geqslant 0\}.$$

(c) f is topologically mixing on J, i.e. for every open set U properly intersecting J there is an integer $n \geqslant 0$ such that $f^n U \supset J$.

We call J a *mixing repeller* for f.

The most frequently encountered examples of mixing repellers are the Julia sets of conformal mappings (see Blanchard (1984) for a survey). If $f(z)$ is a rational function of the (extended) complex plane, for example $f(z) = z^2 + c$, the Julia set (defined to be the closure of the repelling periodic points of f) is, in general, a mixing repeller. Ruelle's (1983) description of dynamical and geometrical properties, such as the Lyapunov exponents and the dimension of the repeller in thermodynamic language, applies in this case. This theory is dependent on the chain rule equality $|(f\circ g)'(z)| = |f'(g(z))||g'(z)|$ which holds for conformal mappings. The plane transformation $f(x_1, x_2) = (x_1^2 - \lambda x_2^2 + a, 2x_1x_2 + b)$ is non-conformal if $\lambda \neq 1$, but nevertheless can have a mixing repeller of 'Julia type'. The standard thermodynamic formalism cannot be applied in this situation since the derivatives merely satisfy an inequality $\|(f\circ g)'(x_1, x_2)\| \leq \|f'(g(x_1, x_2))\| \, \|g'(x_1, x_2)\|$. However, as we shall see, this submultiplicativity is sufficient for a thermodynamic description to be developed.

By standard theory (Bowen 1975, Ruelle 1978) a mixing repeller J has a *Markov partition*, i.e. there exist sets $I_1, \ldots, I_k$, each the closure of their interior (in J), such that $J = \bigcup_{i=1}^{k} I_i$, each $f(I_i)$ is a union of I_j and $\operatorname{int} I_i \cap \operatorname{int} I_j = \varnothing$ if $i \neq j$. A sequence $i_0, i_1, \ldots, i_n$ is called *admissible* if $f(I_{i_j}) \supset I_{i_{j+1}}$ for $0 \leq j \leq n-1$. Given any admissible sequence write $I_{i_0,\ldots,i_n}$ for the *n-cylinder* $\bigcap_{j=0}^{n} f^{-j}(I_{i_j})$. It follows from (1) that $\operatorname{diam} I_{i_0,\ldots,i_n} \leq c\gamma^n$ for some $c > 0$, $\gamma < 1$. Let $\mathscr{C}_n$ denote the collection of admissible n-cylinders. For simplicity of exposition we will assume that I_i and I_j are disjoint for $i \neq j$. The modifications required in the general case are just as in the usual additive situation.

A *subadditive valuation* on M is a sequence of functions $\phi_n : M \to \mathbb{R} (1 \leq n < \infty)$ such that

$$\phi_{m+n}(x) \leq \phi_n(x) + \phi_m(f^n x). \tag{2}$$

We assume a uniform bound and Lipschitz condition

$$|(1/n)\phi_n(x)| \leq M \qquad \text{and} \qquad |(1/n)\phi_n(x) - (1/n)\phi_n(y)| \leq a|x-y|. \tag{3}$$

We also assume that the $\{\phi_n\}$ have bounded variation, i.e. there exists a constant b independent of n such that

$$|\phi_n(x) - \phi_n(y)| \leq b \tag{4}$$

whenever $x, y \in I_j^n$ for some n-cylinder I_j^n. (Conventional thermodynamics uses an additive valuation so that $\phi_n(x) = \phi(x) + \phi(fx) + \ldots + \phi(f^n x)$ for some ϕ. If ϕ is $C^{1+\varepsilon}$ then (3) and (4) certainly hold.)

Let $\mathscr{I}_N$ be the class of (Borel) probability measures on J (or indeed on V) that are invariant under the Nth interate f^N, and let $\mathscr{I} = \bigcup_{N=1}^{\infty} \mathscr{I}_N$. Thus

$$\int g(f^N x)\, \mathrm{d}\nu = \int g(x)\, \mathrm{d}\nu$$

where $\nu \in \mathscr{I}_N$ and g is continuous. The *entropy* of f with respect to $\nu \in \mathscr{I}_1$ exists (see Ruelle 1978, Walters 1982), being defined by

$$h_\nu(f) = \lim_{n\to\infty} -\frac{1}{n} \sum_{\mathscr{C}_n} \nu(I_j^n) \log \nu(I_j^n). \tag{5}$$

As always, summation is over admissible n-cylinders and $0 \log 0 = 0$. $\nu \in \mathscr{I}_1$ is also invariant under f^N and we have

$$h_\nu(f^N) = N h_\nu(f). \tag{6}$$

If $\nu \in \mathscr{I}_N$ then $h_\nu(f^N)$ is defined. Moreover it is easily shown that the limit (5) still exists and satisfies (6). Thus it makes sense to define

$$h_\nu(f) \equiv \frac{1}{N} h_\nu(f^N) = \lim_{n\to\infty} -\frac{1}{n} \sum_{\mathscr{C}_n} \nu(I_j^n) \log \nu(I_j^n) \tag{7}$$

if $\nu \in \mathscr{I}_N$ for any N.

Further, if $\nu \in \mathscr{I}_N$ then

$$\int \phi_{(r+s)N}(x)\,\mathrm{d}\nu \leqslant \int [\phi_{rN}(x) + \phi_{sN}(f^{rN}x)]\,\mathrm{d}\nu = \int \phi_{rN}(x)\,\mathrm{d}\nu + \int \phi_{sN}(x)\,\mathrm{d}\nu.$$

Thus by the usual property of subadditive sequences

$$\lim_{r\to\infty} \frac{1}{rN} \int \phi_{rN}(x)\,\mathrm{d}\nu$$

exists. Since for $1 \leqslant k \leqslant N$

$$\int \phi_{rN+k}(x)\,\mathrm{d}\nu \leqslant \int [\phi_{rN}(x) + \phi_k(f^{rN}x)]\,\mathrm{d}\nu = \int \phi_{rN}(x)\,\mathrm{d}\nu + \int \phi_k(x)\,\mathrm{d}\nu$$

it follows that

$$\lim_{n\to\infty} \frac{1}{n} \int \phi_n(x)\,\mathrm{d}\nu$$

exists if $\nu \in \mathscr{I}_N$ for any N.

In the additive situation, the pressure $P(f, .)$ of f is given by

$$P(f, \phi) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathscr{C}_n} \exp(\phi(x_j) + \phi(fx_j) + \ldots + \phi(f^{n-1}x_j)) \tag{8}$$

where $x_j \in I_j^n$ are chosen to maximise this expression. In the subadditive situation we define the *pressure* as

$$P = P(f, \{\phi_n\}) = \lim_{n\to\infty} \frac{1}{n} \log \sum_{\mathscr{C}_n} \exp(\phi_n(x_j)) \tag{9}$$

where $x_j \in I_j^n$ are always chosen to maximise $\phi_n(x_j)$. (For convenience we write P for the pressure when the arguments are clear.) The existence of this limit is guaranteed by subadditivity. Moreover

$$\begin{aligned}
&\frac{1}{Nr} \log \sum_{\mathscr{C}_{Nr}} \exp(\phi_{Nr}(x_j)) \\
&\qquad \leqslant \frac{1}{Nr} \log \sum_{\mathscr{C}_{Nr}} \exp(\phi_N(x_j) + \phi_N(f^N x_j) + \ldots + \phi_N(f^{(r-1)N} x_j)) \\
&\qquad \leqslant \frac{1}{N} \log \sum_{\mathscr{C}_N} \exp(\phi_N(x_j))
\end{aligned}$$

so, letting $r \to \infty$ and using (8) and (9),

$$P \leqslant \frac{1}{N} P(f^N, \phi_N) \leqslant \frac{1}{N} \log \sum_{\mathscr{C}_N} \exp(\phi_N(x_j)) \tag{10}$$

and letting $N \to \infty$ gives

$$P = \lim_{N\to\infty} \frac{1}{N} P(f^N, \phi_N). \tag{11}$$

Note that by (4) the same value of P is obtained in (9) by taking the x_j to be any points of I_j^n. In particular, choosing x_j as the unique fixed point of f^n in I_j^n it follows that

$$P = \lim_{n\to\infty} \frac{1}{n} \log \sum_{x \in \mathrm{fix} f^n} \exp(\phi_n(x)). \tag{12}$$

For each n, by maximising subject to the constraint $\Sigma_{\mathscr{C}_n} \nu(I_j^n) = 1$, we have

$$\frac{1}{n} \sum_{\mathscr{C}_n} \nu(I_j^n)[-\log \nu(I_j^n) + \phi_n(x_j)] \leqslant \frac{1}{n} \log \sum_{\mathscr{C}_n} \exp(\phi_n(x_j)).$$

For $\nu \in \mathscr{I}_N$, letting $n \to \infty$ gives

$$h_\nu(f) + \lim_{n\to\infty} \frac{1}{n} \int \phi_n(x)\, \mathrm{d}\nu \leqslant P$$

(we use (4) to get the integral). On the other hand, by (3) and the Arzela-Ascoli theorem, there is a subsequence $\{(1/N(i))\phi_{N(i)}(x)\}$ uniformly convergent on J. Given $\varepsilon > 0$, choose $N = N(i)$ such that

$$\sup_{x\in J} \left| \frac{1}{N} \phi_N(x) - \frac{1}{N(j)} \phi_{N(j)}(x) \right| < \varepsilon$$

for $j > i$. By the standard thermodynamical theory (Bowen 1975, Ruelle 1978), there exists an equilibrium measure $\nu \in \mathscr{I}_N$ such that

$$\frac{1}{N} h_\nu(f^N) + \frac{1}{N} \int \phi_N(x)\, \mathrm{d}\nu = \frac{1}{N} P(f^N, \phi_N) \tag{13}$$

so that

$$h_\nu(f) + \lim_{n\to\infty} \frac{1}{n} \int \phi_n(x)\, \mathrm{d}\nu + \varepsilon > \frac{1}{N} P(f^N, \phi_N) \geqslant P.$$

Thus, in the subadditive case we have the *variational principle*

$$\sup_{\nu \in \mathscr{I}} \left(h_\nu(f) + \lim_{n\to\infty} \frac{1}{n} \int \phi_n(x)\, \mathrm{d}\nu \right) = P. \tag{14}$$

In the subadditive case, 'equilibrium measures' for which this supremum is attained are unlikely to exist in general. Taking a weak limit of equilibrium measures for $(1/N)P(f^N, \phi_N)$ yields a probability measure that need not be invariant in any sense, presenting difficulties with the definition of entropy. Nevertheless, we can obtain measures that are 'near Gibbs measures' and that are useful in, for example, the study of Hausdorff dimensions of J (Falconer 1988). Given $\varepsilon > 0$, choose N such that

$$\frac{1}{N} P(f^N, \phi_N) < P + \tfrac{1}{4}\varepsilon \tag{15}$$

and

$$\frac{1}{N} \int \phi_N\, \mathrm{d}\nu < \lim_{n\to\infty} \frac{1}{n} \int \phi_n\, \mathrm{d}\nu + \tfrac{1}{4}\varepsilon.$$

There is a Gibbs measure (Bowen 1975, Ruelle 1978) $\nu \in \mathscr{I}_N$ which satisfies (13) and such that for $0 < a < b$ independent of r

$$a < \nu(I_j^{Nr}) \exp(rP(f^N, \phi_N) - \phi_N(x) - \phi_N(f^N x) - \ldots - \phi_N(f^{(N-1)r} x)) < b \tag{16}$$

for any Nr-cylinder I_j^{Nr} and any $x \in I_j^{Nr}$. By the ergodic theorems of Birkhoff and Kingman (see Walters 1982) there exists a function η such that

$$\frac{1}{r}(\phi_N(x)+\phi_N(f^N x)+\ldots+\phi_N(f^{(N-1)r}x)-\phi_{Nr}(x)) \to \eta(x) \geqslant 0 \tag{17}$$

for ν almost all x, with

$$\int \eta(x)\,\mathrm{d}\nu = \int \phi_N(x)\,\mathrm{d}\nu - \lim_{r\to\infty}\frac{1}{r}\int \phi_{nr}(x)\,\mathrm{d}\nu < \tfrac{1}{4}N\varepsilon.$$

Hence there is a Borel set E with $\nu(E) > \frac{1}{2}$ such that $\eta(x) \leqslant \frac{1}{2}N\varepsilon$ on E; by Egoroff's theorem we may choose E so that convergence in (17) is uniform on E. Thus if $r \geqslant r_0$ and $x \in E \cap I_j^n$ (15)–(17) give

$$\nu(I_j^{Nr}) < b\exp(Nr(-P+\varepsilon)+\phi_{Nr}(x)). \tag{18}$$

Defining a Borel measure by $\nu_1(A) = \nu(A \cap E)$, then $\nu_1(J) > \frac{1}{2}$, and interpolating (18) for cylinders other than Nr-cylinders gives the 'near-Gibbs' property of ν_1:

$$\nu_1(I_j^n) < b_1 \exp[n(-P+\varepsilon)+\phi_n(x)] \tag{19}$$

for all n-cylinders I_j^n and $x \in I_j^n$. (An alternative derivation of such measures using net measures rather than ergodic theory is included in Falconer (1988).) In particular, if $P > 0$, then by choosing $\varepsilon < P$, we have

$$\tfrac{1}{2} < \sum_{\mathscr{C}} \nu_1(I_j^n) \leqslant b_1 \sum_{\mathscr{C}} \exp(\phi_n(x)) \tag{20}$$

for *any* collection of n-cylinders $\mathscr{C}$ that cover J.

Our principal application is closely related to the distribution of Lyapunov exponents of a dynamical system. If S is a contracting linear mapping on $\mathbb{R}^n$, the *singular values* of S, $1 > \alpha_1 \geqslant \alpha_2 \geqslant \ldots \geqslant \alpha_n > 0$ are the positive square roots of the eigenvalues of S^*S or, equivalently, the lengths of the principal semiaxes of the ellipsoids $S(B)$, where B is the unit ball in $\mathbb{R}^n$. The singular value function is defined for $0 < s \leqslant n$ as

$$\psi^s(S) = \alpha_1\alpha_2\ldots\alpha_{m-1}\alpha_m^{s-m+1}$$

where m is the integer such that $m-1 < s \leqslant m$ and, for convenience, as

$$\psi^s(S) = (\alpha_1\alpha_2\ldots\alpha_n)^{s/n}$$

for $s > n$. Then $\psi^s(S)$ is strictly decreasing in s and is submultiplicative, i.e. $\psi^s(S_1S_2) \leqslant \psi^s(S_1)\psi^s(S_2)$ (see Falconer 1988). Thus, with T_x the tangent map of f at x, the valuations $\phi_n^s(x) = \log\psi^s((T_xf^n)^{-1}) = \log\psi^s((T_{f^{n-1}x}f)^{-1}\ldots(T_{fx}f)^{-1}(T_xf)^{-1})$ are subadditive. We may apply the subadditive thermodynamic theory to obtain the pressures

$$P_s \equiv P(f,\{\phi_n^s\}) = \lim_{n\to\infty}\frac{1}{n}\log\sum_{\mathscr{C}_n}\psi^s((T_{x_j}f^n)^{-1} \tag{21}$$

$$= \lim_{n\to\infty}\frac{1}{n}\log\sum_{x\in\mathrm{fix}\,f^n}\psi^s((T_xf^n)^{-1}). \tag{22}$$

Taking local charts of M, the cylinder I_j^n is contained in a parallelepiped of sides at most constant $\times\,\alpha_i$, where α_i are the singular values of $(T_{x_j}f^n)^{-1}$ and $x_j \in I_j^n$. For $s \leqslant n$, I_j^n may be covered by at most constant $\times\,\alpha_1\ldots\alpha_{m-1}\alpha_m^{1-m}$ balls of diameter α_m where

$m-1<s\leqslant m$ (see Falconer 1988, or Falconer and Marsh 1988). It follows that, if $P_s<0$, the Hausdorff dimension of J, dim J, is at most s (as indeed is the box-counting dimension).

Since P_s is strictly decreasing in s, there is a unique $s_0>0$ satisfying

$$P_s=0. \tag{23}$$

In view of (20) it is tempting to hope that $\dim J=s_0$. This is certainly true if f is conformal—we are in the additive situation and (23) reduces to Bowen's formula $P(f, -s\log\|T_xf\|)=0$, which gives dim J (Bowen 1979, Ruelle 1983). However, there are simple examples of non-conformal mappings with $\dim J<s_0$. The difficulty that arises is geometrical rather than dynamical: the cylinders I_j^n may lie close enough to each other for balls straddling a large number of cylinders to provide a more efficient cover in the definition of Hausdorff dimension than by covering each cylinder individually. However, calculations of Bedford (1988) show that the box-counting dimension of J equals s_0 in the important case where $f: S^1\times\mathbb{R}\rightarrow S^1\times\mathbb{R}$ is an expanding mapping preserving the foilation $(\{x\}\times\mathbb{R}: x\in S^1)$. More generally, one might hope that the generalised Bowen formula (23) gives dim J for generic f in some sense. Work by Falconer (1988) and Falconer and Marsh (1988) on piecewise affine transformations implies that dim J satisfies (23) for a very large set of f when J is totally disconnected, in particular for a set of f that is C^1 dense.

By varying the subadditive valuations, for example by looking at $\phi_n(x)=\log t\psi^s((T_xf^n)^{-1}$, the generalised pressures clearly contain much information about the distribution of the Lyapunov exponents.

Note that the subadditive formalism described could be constructed for a shift on an abstract sequence space. However, given the applications we have in mind, we have presented it for a mixing repeller.

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